

# The Hardness of Embedding Grids and Walls

Yijia Chen  
School of Computer Science  
Fudan University  
yijiachen@fudan.edu.cn

Martin Grohe  
Lehrstuhl für Informatik 7  
RWTH Aachen University  
grohe@informatik.rwth-aachen.de

Bingkai Lin  
JST, ERATO, Kawarabayashi Large Graph Project  
National Institute of Informatics  
lin@nii.ac.jp

## Abstract

The dichotomy conjecture for the parameterized embedding problem states that the problem of deciding whether a given graph  $G$  from some class  $\mathcal{K}$  of “pattern graphs” can be embedded into a given graph  $H$  (that is, is isomorphic to a subgraph of  $H$ ) is fixed-parameter tractable if  $\mathcal{K}$  is a class of graphs of bounded tree width and  $W[1]$ -complete otherwise.

Towards this conjecture, we prove that the embedding problem is  $W[1]$ -complete if  $\mathcal{K}$  is the class of all grids or the class of all walls.

## 1 Introduction

The *graph embedding* a.k.a *subgraph isomorphism* problem is a fundamental algorithmic problem, which, as a fairly general pattern matching problem, has numerous applications. It has received considerable attention since the early days of complexity theory (see, e.g., [10, 12, 17, 21]). Clearly, the embedding problem is NP-complete, because the clique problem and the Hamiltonian path or cycle problem are special cases. The embedding problem and special cases like the clique problem or the longest path problem have also played an important role in the development of fixed-parameter algorithms and parameterized complexity theory (see [16]). The problem is complete for the class  $W[1]$  when parameterized by the size of the pattern graph; in fact, the special case of the clique problem may be regarded as the paradigmatic  $W[1]$ -complete problem [8, 9]. On the other hand, interesting special cases such as the longest path and longest cycle problems are fixed-parameter tractable [1, 18]. This immediately raises the question for which pattern graphs the problem is fixed-parameter tractable.

Let us make this precise. An *embedding* from a graph  $G$  to a graph  $H$  is an injective mapping  $f : V(G) \rightarrow V(H)$  such that for all edges  $vw \in E(G)$  we have  $f(v)f(w) \in E(H)$ . For each class  $\mathcal{K}$  of graphs, we consider the following parameterized problem.

$p\text{-EMB}(\mathcal{K})$

*Instance:* Graphs  $G$  (the *pattern graph*) and  $H$  (the *target graph*), where  $G \in \mathcal{K}$ .

*Parameter:*  $|G|$ .

*Problem:* Decide whether there is an embedding from  $G$  to  $H$ .

Plehn and Voigt [19] proved that  $p\text{-EMB}(\mathcal{K})$  is fixed-parameter tractable if  $\mathcal{K}$  is a class of graphs of bounded tree width. No tractable classes  $\mathcal{K}$  of unbounded treewidth are known. The conjecture, which may have been stated in [13] first, is that there are no such classes.

**Dichotomy Conjecture.**  $p\text{-EMB}(\mathcal{K})$  is fixed-parameter tractable if and only if  $\mathcal{K}$  is a class of bounded treewidth and  $W[1]$ -complete otherwise.<sup>1</sup>

Progress towards this conjecture has been slow. Even the innocent-looking case where  $\mathcal{K}$  is the class of complete bipartite graphs had been open for a long time; only recently the third author of this paper proved that it is  $W[1]$ -complete [15].

Before we present our contribution, let us discuss why we expect a dichotomy in the first place. The main reason is that similar dichotomies hold for closely related problems. The first author, jointly with Thurley and Weyer [4], proved the version of the conjecture for the *strong embedding*, or *induced subgraph isomorphism* problem. Building on earlier work by Dalmau, Kolaitis and Vardi [6] as well as joint work with Schwentick and Segoufin [14], the second author [13] proved that the parameterized homomorphism problem  $p\text{-HOM}(\mathcal{K})$  for pattern graphs from a class  $\mathcal{K}$  is fixed-parameter tractable if and only if the cores of the graphs in  $\mathcal{K}$  have bounded tree width and  $W[1]$ -complete otherwise.

Let us remark that there is no P vs. NP dichotomy for the classical (unparameterized) embedding problem; this can easily be proved along the lines of corresponding results for the homomorphism and strong embedding problems using techniques from [2, 4, 13].

## Our contribution

We make further progress towards the Dichotomy Conjecture by establishing hardness for two more natural graph classes of unbounded tree width.

**Theorem 1.1.**  $p\text{-EMB}(\mathcal{K})$  is  $W[1]$ -hard for the classes  $\mathcal{K}$  of all grids and all walls.

See Section 2 and in particular Figure 2 for the definition of grids and walls. Grids and walls are interesting in this context, because they are often viewed as the “generic” graphs of unbounded tree width: by Robertson and Seymour’s [20] Excluded Grid Theorem, a class  $\mathcal{K}$  of graphs has unbounded tree width if and only if all grids (and also all walls) appear as minors of the graphs in  $\mathcal{K}$ .

Just like the hardness result of the embedding problem for the class of all complete bipartite graphs [15], our theorem looks simple and straightforward, but it is not. In fact, we started to work on this right after the hardness for complete bipartite graphs was proved, hoping that we could adapt the techniques to grids. This turned out to be a red herring. The proof we eventually found is closer to the proof of the dichotomy result for the homomorphism problem [13] (also see [3]). The main part of our proof is fairly generic and has

---

<sup>1</sup>There is a minor issue here regarding the computability of the class  $\mathcal{K}$ : if we want to include classes  $\mathcal{K}$  that are not recursively enumerable here then we need the nonuniform notion of fixed-parameter tractability [11].

nothing to do with grids or walls. We prove a general hardness result (Theorem 4.15) for  $p\text{-EMB}(\mathcal{K})$  under the technical condition that the graphs in  $\mathcal{K}$  have “rigid skeletons” and unbounded tree width even after the removal of these skeletons. We think that this theorem may have applications beyond grids and walls.

## Organization of the paper

We introduce necessary notions and notations in Section 2. For some technical reason, we need a colored version  $p\text{-COL-EMB}$  of  $p\text{-EMB}$ . In Section 3 the problem  $p\text{-COL-EMB}$  is shown to be  $W[1]$ -hard on any class of graphs of unbounded treewidth. Then in Section 4, we set up the general framework. In particular, we explain the notion of skeletons, and prove the general hardness theorem. The classes of grids and walls are shown to satisfy the assumptions of this theorem in Section 5. In the final Section 6 we conclude with some open problems.

## 2 Preliminaries

A graph  $G$  consists of a finite set of vertices  $V(G)$  and a set of edges  $E(G) \subseteq \binom{V}{2}$ . Every edge is denoted interchangeably by  $\{u, v\}$  or  $uv$ . We assume familiarity with the basic notions and terminology from graph theory, e.g., degree, path, cycle etc, which can be found in e.g., [7]. By  $\text{dist}^G(u, v)$  we denote the distance between vertices  $u$  and  $v$  in a graph  $G$ , i.e., the length of a shortest path between  $u$  and  $v$ .

Let  $s, t \in \mathbb{N}$ . A  $(s \times t)$ -grid  $G_{s,t}$  has

$$V(G_{s,t}) = [s] \times [t] \quad \text{and} \quad E(G_{s,t}) = \{(i, j)(i', j') \mid |i - i'| + |j - j'| = 1\}.$$

And the wall  $W_{s,t}$  of width  $s$  and height  $t$  is defined by

$$\begin{aligned} V(W_{s,t}) &= \{v_{i,j} \mid i \in [s+1] \text{ and } j \in [t]\} \cup \{v_{i,t+1} \mid i \in [s+1] \text{ and odd } t\} \\ &\quad \cup \{u_{i,j} \mid i \in [s+1] \text{ and } 2 \leq j \leq t\} \cup \{u_{i,t+1} \mid i \in [s+1] \text{ and even } t\}, \\ E(W_{s,t}) &= \{v_{i,1}v_{i+1,1} \mid i \in [s]\} \\ &\quad \cup \{v_{i,t+1}v_{i+1,t+1} \mid i \in [s] \text{ and odd } t\} \\ &\quad \cup \{u_{i,t+1}u_{i+1,t+1} \mid i \in [s] \text{ and even } t\} \\ &\quad \cup \{v_{i,j}u_{i,j}, u_{i,j}v_{i+1,j} \mid i \in [s] \text{ and } 2 \leq j \leq t\} \\ &\quad \cup \{v_{i,j}v_{i,j+1} \mid i \in [s+1] \text{ and odd } j \in [t]\} \\ &\quad \cup \{u_{i,j}u_{i,j+1} \mid i \in [s+1] \text{ and even } j \in [t]\}. \end{aligned}$$

Figure 2 gives two examples.

Let  $G$  and  $H$  be two graphs. A *homomorphism* from  $G$  to  $H$  is a mapping  $h : V(G) \rightarrow V(H)$  such that for every edge  $uv \in E(G)$  we have  $h(u)h(v) \in E(H)$ . If in addition  $h$  is injective, then  $h$  is an *embedding* from  $G$  to  $H$ . A homomorphism from  $G$  to itself is also called an *endomorphism*, and similarly an embedding from  $G$  to itself is an *automorphism*.

A *subgraph*  $G'$  of  $G$  satisfies  $V(G') \subseteq V(G)$  and  $E(G') \subseteq E(G)$ . We say that  $G'$  is a *core* of  $G$  if there is a homomorphism from  $G$  to  $G'$ , and if there is no homomorphism from  $G$  to any proper subgraph of  $G'$ . It is well known that all cores of  $G$  are isomorphic, hence we can speak of *the* core of  $G$ , written  $\text{core}(G)$ .

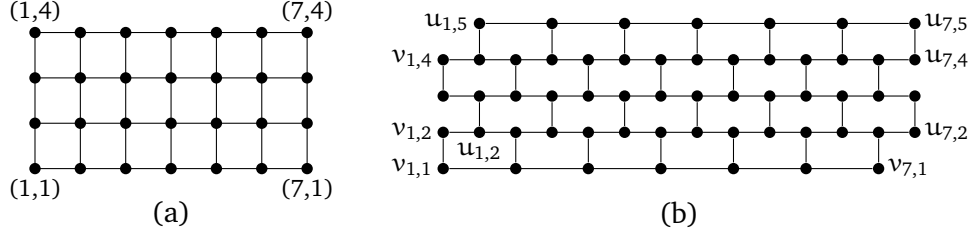


Figure 1: (a) A  $(7 \times 4)$ -grid. (b) A  $(6 \times 4)$ -wall.

Sometimes, we also consider *colored* graphs in which a graph  $G$  is equipped with a coloring  $\chi : V(G) \rightarrow C$  which maps every vertex to a color in the color set  $C$ . We leave it to the reader to generalize the notions of homomorphism, embedding, and core from graphs to colored graphs. One easy but important fact is that if in the colored graph  $(G, \chi)$  every vertex has a distinct color then  $\text{core}(G) = G$ .

The notions of tree decomposition and treewidth are by now standard. In particular,  $\text{tw}(G)$  denotes the treewidth of the graph  $G$ . For a  $(s \times t)$ -grid  $G_{s,t}$ , we have  $\text{tw}(G_{s,t}) = \min\{s, t\}$ , and for a  $(s \times t)$ -wall  $W_{s,t}$ , we have  $\text{tw}(W_{s,t}) = \min\{s, t\} + 1$ . The treewidth of a colored graph  $(G, \chi)$  is the same as the treewidth of the underlying uncolored graph  $G$ , i.e.,  $\text{tw}(G, \chi) = \text{tw}(G)$ .

In a *parameterized problem*  $(Q, \kappa)$  every problem instance  $x \in \{0, 1\}^*$  has a parameter  $\kappa(x) \in \mathbb{N}$  which is computable in polynomial time from  $x$ .  $(Q, \kappa)$  is *fixed-parameter tractable* (FPT) if we can decide for every instance  $x \in \{0, 1\}^*$  whether  $x \in Q$  in time  $f(\kappa(x)) \cdot |x|^{O(1)}$ , where  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a computable function. Thus, FPT plays the role of P in parameterized complexity. On the other hand, the so-called class W[1] is generally considered as a parameterized analog of NP. The precise definition of W[1] is not used in our proofs, so the reader is referred to the standard textbooks, e.g., [8, 11, 5]. Let  $(Q_1, \kappa_1)$  and  $(Q_2, \kappa_2)$  be two parameterized problems. An *fpt-reduction* from  $(Q_1, \kappa_1)$  and  $(Q_2, \kappa_2)$  is a mapping  $R : \{0, 1\}^* \rightarrow \{0, 1\}^*$  such that for every  $x \in \{0, 1\}^*$

- $x \in Q_1 \iff R(x) \in Q_2$ ,
- $R(x)$  can be computed in time  $f(\kappa_1(x)) \cdot |x|^{O(1)}$ , where  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a computable function,
- $\kappa_2(R(x)) \leq g(\kappa_1(x))$ , where  $g : \mathbb{N} \rightarrow \mathbb{N}$  is computable.

Now we state a version of the main result of [13] which is most appropriate for our purpose.

**Theorem 2.1.** *Let  $\mathcal{K}$  be a recursively enumerable<sup>2</sup> class of colored graphs such that for every  $k \in \mathbb{N}$  there is a colored graph  $(G, \chi) \in \mathcal{K}$  whose core has treewidth at least  $k$ . Then  $\text{p-HOM}(\mathcal{K})$  is hard for W[1] (under fpt-reductions).*

### 3 From Homomorphism to Colored Embedding

Let  $\mathcal{K}$  be a class of graphs. We consider the following colored version of the embedding problem for  $\mathcal{K}$ .

<sup>2</sup>If  $\mathcal{K}$  is not recursively enumerable, there is still a “non-uniform” hardness result. See [13] or a discussion.

<b>p-COL-EMB(<math>\mathcal{K}</math>)</b>	
<i>Instance:</i>	Two graphs $G$ and $H$ with $G \in \mathcal{K}$ , and a function $\chi : V(H) \rightarrow V(G)$ .
<i>Parameter:</i>	$ G $ .
<i>Problem:</i>	Decide whether there is an embedding $h$ from $G$ to $H$ such that $\chi(h(v)) = v$ for every $v \in V(G)$ .

Thus, in the p-COL-EMB( $\mathcal{K}$ ) problem we partition the vertices of  $H$  and associate one part with each vertex of  $G$ . Then we ask for an embedding where each vertex  $G$  is mapped to its part.

**Lemma 3.1.** *Let  $\mathcal{K}$  be a recursively enumerable class of graphs with unbounded treewidth. Then p-COL-EMB( $\mathcal{K}$ ) is hard for W[1].*

*Proof.* Let  $G$  be a graph. We expand it with the trivial coloring  $\chi_G : V(G) \rightarrow V(G)$  defined by  $\chi_G(v) := v$  for every  $v \in V(G)$ . It is easy to see that  $\text{core}(G, \chi_G) = (G, \chi_G)$  and hence  $\text{tw}(\text{core}(G, \chi_G)) = \text{tw}(G, \chi_G) = \text{tw}(G)$ . Therefore, the class

$$\mathcal{K}^* := \{(G, \chi_G) \mid G \in \mathcal{K}\}$$

satisfies the conditions in Theorem 2.1. Hence, p-HOM( $\mathcal{K}^*$ ) is W[1]-hard, and it suffices to show p-HOM( $\mathcal{K}^*$ ) can be reduced to p-COL-EMB( $\mathcal{K}$ ) by an fpt-reduction.

Let  $(G, \chi_G) \in \mathcal{K}^*$  with  $G \in \mathcal{K}$  and  $(H, \chi_H)$  be a colored graph. We construct a graph  $P$  with

$$\begin{aligned} V(P) &:= \{(u, v) \in V(G) \times V(H) \mid \chi_H(v) = u\}, \\ E(P) &:= \{(u_1, v_1)(u_2, v_2) \mid (u_1, v_1), (u_2, v_2) \in V(P), \\ &\quad u_1 u_2 \in E(G), \text{ and } v_1 v_2 \in E(H)\}. \end{aligned}$$

Moreover let  $\chi : V(P) \rightarrow V(G)$  be given by

$$\chi(u, v) := u$$

for every  $(u, v) \in V(P)$ . It is easy to verify that

$$\begin{aligned} &\text{there is a homomorphism from } (G, \chi_G) \text{ to } (H, \chi_H) \\ \iff &\text{there is an embedding } h \text{ from } G \text{ to } P \text{ with } \chi(h(v)) = v \text{ for every } v \in V(G). \end{aligned}$$

□

## 4 Frames and Skeletons

Let  $G$  be a graph and  $D \subseteq V(G)$  such that the degree of every  $v \in D$  is at most 2, i.e.,  $\deg^G(v) \leq 2$ . For every  $u, v \in V(G) \setminus D$  we say they are *close* (with respect to  $D$ ) if there is a

path in  $G$  between  $u$  and  $v$  whose internal vertices are all in  $D$ . We define  $G/D$  as the graph given by

$$\begin{aligned} V(G/D) &:= V(G) \setminus D, \\ E(G/D) &:= \{uv \mid u, v \in V(G) \setminus D, u \neq v, \text{ and they are close}\}. \end{aligned}$$

Let  $u \in D$ . We say that  $u$  is *associated with a vertex*  $v \in V(G/D)$  if  $u$  is on a path in  $G$  between  $v$  and a vertex  $w \in D$  with  $\deg^G(w) = 1$  whose internal vertices are all in  $D$ . Similarly,  $u$  is *associated with some edge*  $e = vw \in E(G/D)$  if  $u$  is on a path in  $G$  between  $v$  and  $w$  whose internal vertices are all in  $D$ . It should be clear that  $u$  can only be associated with a unique vertex or a unique edge in  $G/D$ , and not both. Furthermore, some  $w \in D$  might not be associated with any vertex or edge; this happens precisely to all  $w$  on a path or cycle with all vertices in  $D$ .

To simplify presentation, from now on we fix a graph  $G$ .

**Definition 4.1.** A set  $F \subseteq V(G)$  is a *frame* for  $G$  if every endomorphism  $h$  of  $G$  with  $F \subseteq h(V(G))$  is surjective.

**Remark 4.2.** Let  $F$  be a frame for  $G$

- (1) If  $F = \emptyset$ , then  $G$  is a core.
- (2) Any endomorphism  $h$  with  $F \subseteq h(V(G))$  is an automorphism of  $G$ , since  $G$  is finite.
- (3) Let  $F' \subseteq V(G)$  with  $F \subseteq F'$ . Then  $F'$  is also a frame for  $G$ .
- (4)  $V(G)$  is a frame for  $G$ .

**Definition 4.3.** Let  $F, D \subseteq V(G)$  such that

- (S1)  $F$  is a frame for  $G$ ,
- (S2)  $F \cap D = \emptyset$ ,
- (S3) for every  $v \in D$

$$\deg^{G \setminus F}(v) = |\{u \in V \mid u \notin F \text{ and } \{u, v\} \in E\}| \leq 2.$$

Then we call  $\mathcal{S} = (F, D)$  a *skeleton* of  $G$ .

**Example 4.4.** Consider the grid  $G_{7,8}$ . Lemma 5.5 in Section 5 implies that it has a skeleton  $(F, D)$  with

$$\begin{aligned} F &= \{(i, j) \mid i \in \{1, 2, 6, 7\} \text{ or } j \in \{1, 7, 8\}\} \cup \{(4, 2j) \mid j \in [3]\} \quad \text{and} \\ D &= \{(2i + 1, 2j) \mid i \in [2] \text{ and } j \in [3]\} \cup \{(2i, 2j + 1) \mid i \in \{2\} \text{ and } j \in [2]\}, \end{aligned}$$

as shown in Figure 2.

**Definition 4.5.** Let  $\mathcal{S} = (F, D)$  be a skeleton of  $G$ . For every graph  $H$  and every mapping

$$\chi : V(H) \rightarrow V(G) \setminus (F \cup D),$$

we construct a product graph  $P = P(G, \mathcal{S}, H, \chi)$  as follows.

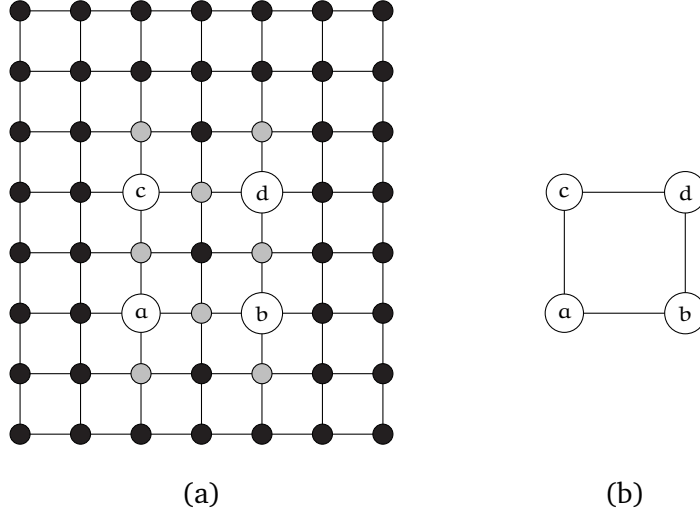


Figure 2: (a) A skeleton for  $G_{7,8}$ , where  $F$  is the set of black vertices and  $D$  the set of light gray vertices. (b) The graph  $(G \setminus F)/D$ .

(P1) The vertex set is  $V(P) := \bigcup_{i \in [4]} V_i$  with

$$\begin{aligned}
 V_1 &= \{(u, a) \mid u \in V(G) \setminus (F \cup D) \text{ and } a \in V(H) \text{ with } \chi(a) = u\}, \\
 V_2 &= \{(u, u) \mid u \in F \text{ or } (u \in D \text{ without being associated} \\
 &\quad \text{with any vertex or edge in } (G \setminus F)/D)\}, \\
 V_3 &= \{(u, v_{u,a}) \mid u \in D, a \in V(H), \text{ and } \chi(a) = v \\
 &\quad \text{with } u \text{ being associated with } v \text{ in } (G \setminus F)/D\}, \\
 V_4 &= \{(u, v_{u,e}) \mid u \in D, e = \{a, b\} \in E(H), \chi(a) = v, \text{ and } \chi(b) = w \\
 &\quad \text{with } u \text{ being associated with } \{v, w\} \text{ in } (G \setminus F)/D\}.
 \end{aligned}$$

Note that in the definition of  $V_3$  and  $V_4$  all  $v_{u,a}$  and  $v_{u,e}$  are fresh elements.

(P2) The edge set is  $E(P) := \bigcup_{1 \leq i \leq j \leq 4} E_{ij}$  with

$$\begin{aligned}
 E_{11} &= \{(u, a)(v, b) \mid (u, a), (v, b) \in V_1, uv \in E(G), \text{ and } ab \in E(H)\}, \\
 E_{12} &= \{(u, a), (v, v) \mid (u, a) \in V_1, (v, v) \in V_2, \text{ and } uv \in E(G)\}, \\
 E_{13} &= \{(u, a)(v, v_{v,a}) \mid (u, a) \in V_1, (v, v_{v,a}) \in V_3, \text{ and } uv \in E(G)\}, \\
 E_{14} &= \{(u, a)(v, v_{v,e}) \mid (u, a) \in V_1, (v, v_{v,e}) \in V_4, uv \in E(G), \text{ and } a \in e\}, \\
 E_{22} &= \{(u, u)(v, v) \mid (u, u), (v, v) \in V_2 \text{ and } uv \in E(G)\}, \\
 E_{23} &= \{(u, u)(v, v_{v,a}) \mid (u, u) \in V_2, (v, v_{v,a}) \in V_3, \text{ and } uv \in E(G)\}, \\
 E_{24} &= \{(u, u)(v, v_{v,e}) \mid (u, u) \in V_2, (v, v_{v,e}) \in V_4, \text{ and } uv \in E(G)\}, \\
 E_{33} &= \{(u, v_{u,a})(v, v_{v,a}) \mid (u, v_{u,a}), (v, v_{v,a}) \in V_3 \text{ and } uv \in E(G)\}, \\
 E_{34} &= \emptyset, \\
 E_{44} &= \{(u, v_{u,e})(v, v_{v,e}) \mid (u, v_{u,e}), (v, v_{v,e}) \in V_4 \text{ and } uv \in E(G)\}.
 \end{aligned}$$

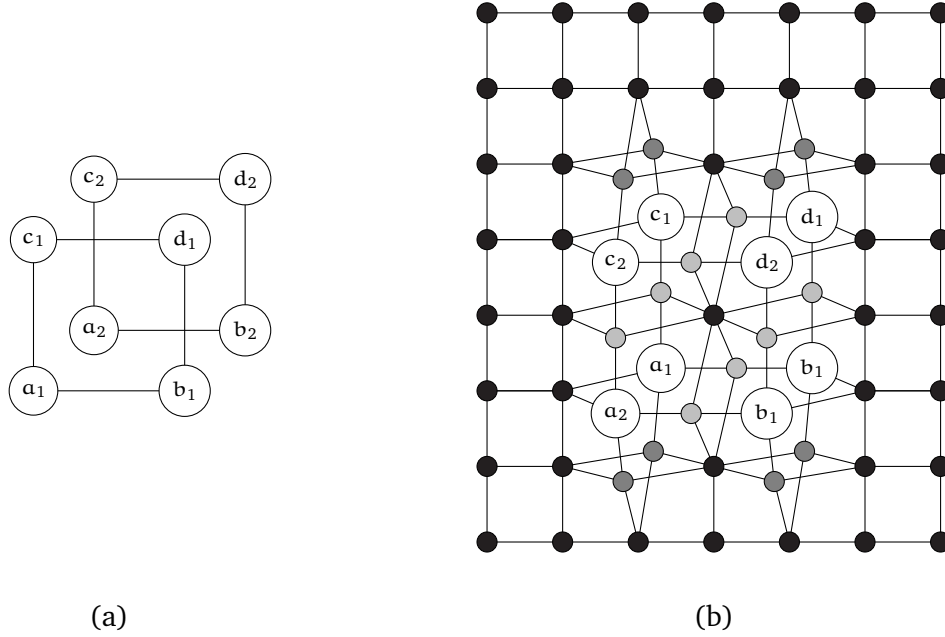


Figure 3:

**Example 4.6.** Let  $H$  be the graph in Figure 3 (a). Moreover, we color every  $a_i$  with  $\chi(a_i) := a$ , every  $b_i$  with  $\chi(b_i) := b$ , and so on. We consider the grid  $G_{7,8}$  and the skeleton  $\mathcal{S} = (F, D)$  defined in Example 4.4. Then Figure 3 (b) is the product  $P = P(G, \mathcal{S}, H, \chi)$ . In particular,  $V_1$  is the set of white vertices,  $V_2$  is the set of black vertices,  $V_3$  is the set of gray vertices, and  $V_4$  is the set of remaining light gray vertices. To make the picture less cluttered, we label the vertex  $(a, a_i)$  by  $a_i$  etc. in  $P$ .

In Definition 4.5 the reader might notice that in each pair  $(u, a)$ ,  $(u, u)$ ,  $(u, v_{u,a})$ , or  $(u, v_{u,e})$  the first coordinate is uniquely determined by the second coordinate. Thus:

**Lemma 4.7.** *Let  $h : V(G) \rightarrow V(P)$  be injective, e.g.,  $h$  is an embedding from  $G$  to  $P$ . Then the mapping  $\pi_2 \circ h$  is injective, too. Here  $\pi_2(u, z) = z$  for every  $(u, z) \in V(P)$  is the projection on the second coordinate.*

**Lemma 4.8.**  *$\pi_1$  is homomorphism from  $P$  to  $G$ , where  $\pi_1(u, z) = u$  for every  $(u, z) \in V(P)$  is the projection on the first coordinate.*

*Proof.* Observe that by (P2) for every edge  $(u, w)(v, z) \in E(P)$  we have  $uv \in E(G)$ . □

**Lemma 4.9.** *Let  $h$  be a homomorphism from  $G$  to  $P$ . Then the mapping  $\pi_1 \circ h$  is an endomorphism of  $G$ . Moreover, if*

$$\{(u, u) \mid u \in F\} \subseteq h(V(G)),$$

*then  $\pi_1 \circ h$  is an automorphism of  $G$ .*

*Proof.* By Lemma 4.8  $\pi_1 \circ h$  is an endomorphism of  $G$ . If  $\{(u, u) \mid u \in F\} \subseteq h(V(G))$ , then  $F \subseteq \pi_1 \circ h(V(G))$ . Since  $F$  is a frame,  $\pi_1 \circ h$  has to be surjective. □



**Lemma 4.10.** *Let  $h$  be a homomorphism from  $G$  to  $P$  such that  $\pi_1 \circ h$  is an automorphism of  $G$ . Then there is a homomorphism  $\bar{h}$  from  $(G \setminus F)/D$  to  $H$  such that  $\chi(\bar{h}(v)) = v$  for every  $v \in V(G) \setminus (F \cup D)$ . Note this implies that  $\bar{h}$  is an embedding.*

*Proof.* Let  $\rho := \pi_1 \circ h$ . By assumption  $\rho$  is an automorphism of  $G$ , hence so is  $\rho^{-1}$ . Thus  $h \circ \rho^{-1}$  is a homomorphism from  $G$  to  $P$  with

$$\pi_1 \circ (h \circ \rho^{-1}) = (\pi_1 \circ h) \circ \rho^{-1} = (\pi_1 \circ h) \circ (\pi_1 \circ h)^{-1} = \text{id}.$$

Hence for every  $u \in V(G)$  there is a  $w$  such that

$$h \circ \rho^{-1}(u) = (u, w) \tag{1}$$

Let

$$\bar{h} := \pi_2 \circ h \circ \rho^{-1}.$$

We claim that  $\bar{h}$  restricted to  $V(G) \setminus (F \cup D)$  is the desired homomorphism from  $(G \setminus F)/D$  to  $H$ .

Let  $u \in V(G) \setminus (F \cup D)$ . By the definition of  $V(P)$  in (P1) and (1) we have  $\bar{h}(u) \in V_1$ , and thus by the definition of  $V_1$ ,

$$\bar{h}(u) \in V(H) \quad \text{with} \quad \chi(\bar{h}(u)) = u.$$

Next, let  $uv \in E((G \setminus F)/D)$ . We have to show  $\bar{h}(u)\bar{h}(v) \in E(H)$ . By the definition of  $(G \setminus F)/D$  there is a path

$$u = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k = v$$

in  $G \setminus F$  with  $k \geq 2$  and all  $v_i \in D$  for  $1 < i < k$ . If  $k = 2$ , then  $uv \in E(G)$ . We can conclude

$$\{(u, \bar{h}(u)), (v, \bar{h}(v))\} = \{h \circ \rho^{-1}(u), h \circ \rho^{-1}(v)\} \in E(P),$$

because  $h \circ \rho^{-1}$  is a homomorphism. Then  $\{\bar{h}(u), \bar{h}(v)\} \in E(H)$  follows directly from the definition of  $E_{11}$  in (P2). So assume  $k > 2$ . Again by (1) and (P1) for some pairwise distinct  $a, b \in V(H)$  and  $w_2, \dots, w_{k-1}$

$$\begin{aligned} h \circ \rho^{-1}(u) &= (u, a), \\ h \circ \rho^{-1}(v_2) &= (v_2, w_2), \\ &\vdots \\ h \circ \rho^{-1}(v_{k-1}) &= (v_{k-1}, w_{k-1}), \\ h \circ \rho^{-1}(v) &= (v, b). \end{aligned}$$

As every  $v_i$  is associated with  $\{u, v\}$ , there are  $e_2, \dots, e_{k-1} \in E(H)$  with  $w_i = \mathbf{v}_{v_i, e_i}$  by the definition of  $V_4$  in (P1). Since  $h \circ \rho^{-1}$  is a homomorphism from  $G$  to  $P$ ,

$$\begin{aligned} \{(u, a), (v_2, \mathbf{v}_{v_2, e_2})\} &\in E(P), \\ \{(v_2, \mathbf{v}_{v_2, e_2}), (v_3, \mathbf{v}_{v_3, e_3})\} &\in E(P), \\ &\vdots \\ \{(v_{k-2}, \mathbf{v}_{v_{k-2}, e_{k-2}}), (v_{k-1}, \mathbf{v}_{v_{k-1}, e_{k-1}})\} &\in E(P), \\ \{(v_{k-1}, \mathbf{v}_{v_{k-1}, e_{k-1}}), (v, b)\} &\in E(P). \end{aligned}$$

Then by the definition of  $E_{44}$  in (P2), we conclude  $e_2 = \cdots = e_{k-1}$ . Finally, the definition of  $E_{14}$  implies that  $e_2 = \{a, b\}$ , i.e.,  $\{\bar{h}(u), \bar{h}(v)\} \in E(H)$ .  $\square$

**Lemma 4.11.** *If there is an embedding  $\bar{h}$  from  $(G \setminus F)/D$  to  $H$  with  $\chi(\bar{h}(v)) = v$  for every  $v \in V(G) \setminus (F \cup D)$ , then there is an embedding from  $G$  to  $P$ .*

*Proof.* We define a mapping  $h : V(G) \rightarrow V(P)$  and show that it is an embedding.

- For  $u \in V(G) \setminus (F \cup D)$  let  $h(u) := (u, \bar{h}(u))$ , which is well defined by  $\chi(\bar{h}(u)) = u$ .
- For  $u \in F$  let  $h(u) := (u, u)$ .
- For  $u \in D$  without being associated with any vertex or edge in  $(G \setminus F)/D$  let  $h(u) := (u, u)$ .
- Let  $u \in D$  be associated with a (unique)  $v \in V((G \setminus F)/D)$ . We set  $h(u) := (u, v_{u, \bar{h}(v)})$ .
- Let  $u \in D$  be associated with a (unique)  $vw \in E((G \setminus F)/D)$ . We set  $h(u) := (u, v_{u, \bar{h}(v)} \bar{h}(w))$ .

The injectivity of  $h$  is trivial. To see that it is a homomorphism, let  $uv \in E(G)$  and we need to establish  $h(u)h(v) \in E(P)$ .

- Assume  $u, v \in V(G) \setminus (F \cup D)$ . Then  $uv \in E(G)$  implies  $uv \in E((G \setminus F)/D)$ , and as  $\bar{h}$  is a homomorphism from  $(G \setminus F)/D$  to  $H$ , it follows that  $\bar{h}(u)\bar{h}(v) \in E(H)$ . So by the definition of  $E_{11}$  in (P1) we conclude  $(u, \bar{h}(u))(v, \bar{h}(v)) \in E(P)$ .
- Let  $u \in V(G) \setminus (F \cup D)$  and  $v \in D$ . Furthermore, assume that  $v$  is associated with an edge  $wz \in E((G \setminus F)/D)$ . Hence,  $h(u) = (u, \bar{h}(u))$  and  $h(v) = (v, v_{v, \bar{h}(w)} \bar{h}(z))$ . Recall  $uv \in E(G)$ , therefore  $u = w$  or  $u = z$ . Then,  $\bar{h}(u) \in \{\bar{h}(w), \bar{h}(z)\}$ , and the definition of  $E_{14}$  in (P2) implies that  $h(u)h(v) \in E(P)$ .
- Assume both  $u, v \in D$  and they are associated with some edges  $e_1, e_2 \in E((G \setminus F)/D)$ . Then  $e_1 = e_2$  by  $uv \in E(G)$ , and  $h(u)h(v) \in E(P)$  follows from the definition of  $E_{44}$  in (P2).
- All the remaining cases are similar and easy.

□

**Definition 4.12.** A skeleton  $\mathcal{S} = (F, D)$  is *rigid* if for every graph  $H$ , every  $\chi : V(H) \rightarrow V(G) \setminus (F \cup D)$ , and every embedding  $h$  from  $G$  to  $P = P(G, \mathcal{S}, H, \chi)$ , it holds that  $\{(u, u) \mid u \in F\} \subseteq h(V(G))$ .

**Proposition 4.13.** *There is an algorithm which lists all rigid skeletons of an input graph  $G$ .*

*Proof.* Let  $G$  be a graph and  $F, D \subseteq V(G)$ . Clearly it is decidable whether  $\mathcal{S} = (F, D)$  is a skeleton by Definition 4.3. Moreover, we observe that  $\mathcal{S}$  is *not* a rigid skeleton if and only if there is graph  $H$ , a mapping  $\chi : V(H) \rightarrow V(G) \setminus (F \cup D)$ , and an embedding  $h$  from  $G$  to  $P = P(G, \mathcal{S}, H, \chi)$  such that

$$\{(u, u) \mid u \in F\} \not\subseteq h(V(G)). \quad (2)$$

We define a set

$$\begin{aligned} X = & \{a \in V(H) \mid (u, a) \in h(G) \text{ for some } u \in V(G) \setminus (F \cup D)\} \\ & \cup \{a \in V(H) \mid (u, v_{u, a}) \in h(G) \text{ for some } u \in D\} \\ & \cup \{a, b \in V(H) \mid (u, v_{u, ab}) \in h(G) \text{ for some } u \in D\}. \end{aligned}$$

It is routine to verify that  $h$  is an embedding from  $G$  to  $P' = P(G, \mathcal{S}, H[X], \chi|_X)$  such that (2) also holds. Hence, the induced subgraph  $H[X]$  with the coloring  $\chi|_X$  also witnesses that  $\mathcal{S}$  is not rigid. Observe that  $|X| \leq 2|V(G)|$ .

Therefore, to list all the rigid skeletons of  $G$ , we enumerate all pairs  $\mathcal{S} = (F, D)$ ,

- check whether  $\mathcal{S}$  is a skeleton,
- and if so, then check whether it is rigid by going through all graphs on the vertex set  $[n]$  with  $n \leq 2|V(G)|$ .

□

**Definition 4.14.** A class  $\mathcal{K}$  of graphs is *rich* if for every  $k \in \mathbb{N}$  there is a graph  $G \in \mathcal{K}$  such that  $G$  has a rigid skeleton  $(F, D)$  with

$$\text{tw}((G \setminus F)/D) \geq k. \quad (3)$$

**Theorem 4.15.** *Let  $\mathcal{K}$  be a recursively enumerable and rich class of graphs. Then  $\text{p-EMB}(\mathcal{K})$  is hard for  $W[1]$ .*

*Proof.* We define a sequence of graphs  $G_1, G_2, \dots$  and sets  $F_i, D_i \subseteq V(G_i)$  as follows. For every  $i \in \mathbb{N}$  we enumerate graphs  $G$  in the class  $\mathcal{K}$  one by one. For every  $G$  we list all the rigid skeletons  $(F, D)$  of  $G$  by Proposition 4.13. Then we check whether there is such a rigid skeleton  $(F, D)$  satisfying (3). If so, we let  $G_i := G$ ,  $(F_i, D_i) := (F, D)$ , and define

$$G_i^* := (G_i \setminus F_i)/D_i.$$

By our assumption,  $G_i$  will be found eventually, and  $G_i^*$  is well defined and computable from  $G_i$ . It follows that the class

$$\mathcal{K}^* := \{G_i^* \mid i \in \mathbb{N}\}$$

is recursively enumerable and has unbounded treewidth.

By Lemma 3.1, we conclude that  $\text{p-COL-EMB}(\mathcal{K}^*)$  is  $W[1]$ -hard. Hence it suffices to give an fpt-reduction from  $\text{p-COL-EMB}(\mathcal{K}^*)$  to  $\text{p-EMB}(\mathcal{K})$ . Let  $G_i^* \in \mathcal{K}^*$ . Thus  $G_i^* = (G_i \setminus F_i)/D_i$  for the rigid skeleton  $\mathcal{S}_i = (F_i, D_i)$ . Then for every graph  $H$  and  $\chi : V(H) \mapsto V(G_i^*)$  we claim that

$$\begin{aligned} \text{there is an embedding } h \text{ from } G_i^* \text{ to } H \text{ with } \chi(h(v)) = v \text{ for every } v \in V(G_i^*) \\ \iff \text{there is an embedding from } G_i \text{ to } P(G_i, \mathcal{S}_i, H, \chi). \end{aligned}$$

The direction from left to right is by Lemma 4.11. The other direction follows from the rigidity of  $\mathcal{S}_i$ , Lemma 4.9, and Lemma 4.10. □

## 5 Grids and Walls

In this section we show that the classes of grids and walls are rich. More precisely:

**Proposition 5.1.** *Let  $\mathcal{K}$  be a class of graphs.*

- (i) *If every  $k \in \mathbb{N}$  there exists a grid  $G_{s,t} \in \mathcal{K}$  with  $\min\{s, t\} \geq k$ . Then  $\mathcal{K}$  is rich.*

(ii) If every  $k \in \mathbb{N}$  there exists a wall  $W_{s,t} \in \mathcal{K}$  with  $\min\{s, t\} \geq k$ . Then  $\mathcal{K}$  is rich.

Now the following more general version of Theorem 1.1 is an immediate consequence of Theorem 4.15 and Proposition 5.1.

**Theorem 5.2.** *Let  $\mathcal{K}$  be a recursively enumerable class of graphs. Then  $\text{p-EMB}(\mathcal{K})$  is  $W[1]$ -hard if one of the following conditions is satisfied.*

1. *For every  $k \in \mathbb{N}$  there exists a grid  $G_{k_1, k_2} \in \mathcal{K}$  with  $\min\{k_1, k_2\} \geq k$ .*
2. *For every  $k \in \mathbb{N}$  there exists a wall  $W_{k_1, k_2} \in \mathcal{K}$  with  $\min\{k_1, k_2\} \geq k$ .*

## 5.1 The richness of grids

**Lemma 5.3.** *Let  $s, t \in \mathbb{N}$ . Then the set of the four corner vertices*

$$F := \{(1, 1), (s, 1), (1, t), (s, t)\}$$

*is a frame for  $G_{s,t}$ .*

*Proof.* Let  $h$  be an endomorphism of  $G_{s,t}$  with  $F \subseteq h(G_{s,t})$ .

*Claim 1.*  $h^{-1}(F) = F$ . More precisely,

$$h^{-1}(\{(1, 1), (s, t)\}) = \{(1, 1), (s, t)\} \quad \text{and} \quad h^{-1}(\{(1, t), (s, 1)\}) = \{(1, t), (s, 1)\},$$

or

$$h^{-1}(\{(1, 1), (s, t)\}) = \{(1, t), (s, 1)\} \quad \text{and} \quad h^{-1}(\{(1, t), (s, 1)\}) = \{(1, 1), (s, t)\}.$$

*Proof of the claim.* As  $F \subseteq h(G_{s,t})$ ,

$$\{u, v\} := h^{-1}(\{(1, 1), (s, t)\}).$$

is well defined. Clearly

$$\text{dist}^{G_{s,t}}(u, v) \geq \text{dist}^{G_{s,t}}(h(u), h(v)) = \text{dist}^{G_{s,t}}((1, 1), (s, t)) = s + t - 2,$$

since  $h$  is a homomorphism. But this is only possible if

$$\{u, v\} = \{(1, 1), (s, t)\} \quad \text{or} \quad \{u, v\} = \{(1, t), (s, 1)\}.$$

By the same reasoning,

$$h^{-1}(\{(1, t), (s, 1)\}) = \{(1, t), (s, 1)\} \quad \text{or} \quad h^{-1}(\{(1, t), (s, 1)\}) = \{(1, 1), (s, t)\}.$$

The claim then follows easily. +

So without loss of generality we can assume that  $h(u) = u$  for every  $u \in F$ .

*Claim 2.* Let  $u, v \in V(G_{s,t})$ . If for every  $w \in F$  we have

$$\text{dist}^{G_{s,t}}(u, w) \geq \text{dist}^{G_{s,t}}(v, w),$$

then  $u = v$ .

*Proof of the claim.* Routine. ⊥

Now we are ready to show that  $h(u) = u$  for every  $u \in V(G_{s,t})$ . Note that for every  $w \in F$

$$\begin{aligned} \text{dist}^{G_{s,t}}(u, w) &\geq \text{dist}^{G_{s,t}}(h(u), h(w)) && (h \text{ is a homomorphism}) \\ &= \text{dist}^{G_{s,t}}(h(u), w) && (\text{our assumption } h(w) = w). \end{aligned}$$

Claim 2 implies that  $h(u) = u$ . □

**Definition 5.4.** Let  $s, t \in \mathbb{N}$  with  $s \geq 5$  and  $t \geq 6$ . Set

$$k_1 := \left\lfloor \frac{s-1}{2} \right\rfloor \quad \text{and} \quad k_2 := \left\lfloor \frac{t-2}{2} \right\rfloor. \quad (4)$$

Then we define  $\mathcal{S}_{s,t} := (F, D)$  with

$$\begin{aligned} F &= \{(i, j) \mid (i \in [2] \text{ or } 2k_1 \leq i \leq s) \text{ or } (j = 1 \text{ or } 2k_2 < j \leq t)\} \\ &\quad \cup \{(2i, 2j) \mid i \in [k_1] \text{ and } j \in [k_2]\}, \\ D &= \{(2i+1, 2j) \mid i \in [k_1-1] \text{ and } j \in [k_2]\} \\ &\quad \cup \{(2i, 2j+1) \mid 1 < i < k_1 \text{ and } j \in [k_2-1]\}. \end{aligned}$$

See Figure 2 for  $\mathcal{S}_{7,8}$ .

In the following we fix some  $s, t \in \mathbb{N}$  with  $s \geq 5$  and  $t \geq 6$ .

**Lemma 5.5.**  $\mathcal{S}_{s,t}$  is a skeleton of the grid  $G_{s,t}$ .

*Proof.* The condition (S1) in Definition 4.3 follows from Lemma 5.3 and Remark 4.2 (3). The conditions (S2) and (S3) are immediate. □

**Remark 5.6.** The frame in Definition 5.4 contains much more vertices than what is required in Lemma 5.3. It will become evident for our proof that  $F$  needs to contain top two rows, leftmost two columns, and rightmost two columns in the grid.

The next result is routine.

**Lemma 5.7.**  $(G_{s,t} \setminus F)/D$  is a  $((\lfloor (s-1)/2 \rfloor - 1) \times (\lfloor (t-2)/2 \rfloor - 1))$ -grid, where  $(F, D) = \mathcal{S}_{s,t}$ . Hence

$$\text{tw}((G_{s,t} \setminus F)/D) = \min \left\{ \left\lfloor \frac{s-1}{2} \right\rfloor, \left\lfloor \frac{t-2}{2} \right\rfloor \right\} - 1.$$

**Proposition 5.8.**  $\mathcal{S}_{s,t}$  is a rigid skeleton of the grid  $G_{s,t}$ .

The proof is very technical. We proceed in several steps to improve the readability.

Note that the shape of  $F$  is slightly different depending on the parity of  $s$  and  $t$ . Without loss of generality, we assume that  $s$  is odd and  $t$  is even as in Example 4.4 and Example 4.6. Therefore in (4) we have  $k_1 = (s-1)/2$  and  $k_2 = t/2 - 1$ .

We fix a graph  $H$ , a mapping  $\chi : V(H) \rightarrow V(G_{s,t}) \setminus (F \cup D)$ , and an embedding  $h$  from  $G_{s,t}$  to  $P = P(G_{s,t}, \mathcal{S}_{s,t}, H, \chi)$ . Moreover, let

$$F_1 := \{(2i, 2j) \mid i \in [k_1] \text{ and } j \in [k_2 + 1]\}.$$

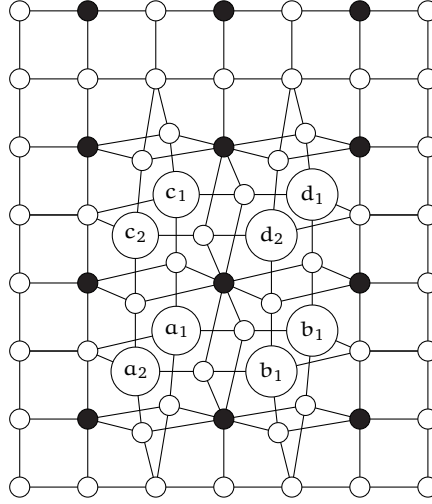


Figure 4: The set  $C$  consists of those black vertices.

**Lemma 5.9.** *Let*

$$C := \{(u, u) \mid u \in F_1\}$$

(which is shown in Figure 4 for our running example).

(C1) Every 4-cycle in  $P$  must contain a vertex in  $C$ .

(C2) Every pair of vertices in  $C$  has even distance in  $P$ .

*Proof.* Recall that  $\pi_1$  is a homomorphism from  $P$  to  $G_{s,t}$  with  $\pi_1(u, z) = u$  for every  $(u, z) \in V(P)$ . Obviously,  $\pi_1(C) = F_1$ . Since  $P$  and  $G_{s,t}$  have no loops, every homomorphism from  $P$  to  $G_{s,t}$  preserves the parity of walk length. In particular, if  $(u_1, z_1), (u_2, z_2) \in V(P)$  have odd distance in  $P$ , then  $u_1$  and  $u_2$  must have a walk with odd length in  $G_{s,t}$ . It is not hard to see that (C2) follows from the fact that every walk for a pair of vertices in  $F_1$  has even length in  $G_{s,t}$ .

To prove (C1), we observe that every 4-cycle in  $G_{s,t}$  must contain a vertex in  $F_1$ . If a 4-cycle  $X$  in  $P$  is mapped to a 4-cycle in  $G_{s,t}$  by  $\pi_1$ , then  $X$  must contain a vertex in  $C$  because  $\pi_1^{-1}(F_1) = C$ . So it suffices to argue that no 4-cycle is mapped to an edge under  $\pi_1$ . Suppose a 4-cycle  $X$  in  $P$  is mapped to an edge under  $\pi_1$ . Then there must exist  $u, u' \in V(G_{s,t})$  such that  $V(X) = \{(u, z_1), (u', z_2), (u, z_3), (u', z_4)\}$  and it forms a 4-cycle

$$X : (u, z_1) \rightarrow (u', z_2) \rightarrow (u, z_3) \rightarrow (u', z_4) \rightarrow (u, z_1).$$

Recall that the vertex set of  $P$  is partitioned into 4 subsets  $V_1, V_2, V_3$ , and  $V_4$ .

- First, we note that for every vertex  $(v, b) \in V_2$ , there is no vertex  $(v, b') \in V(P)$  with  $b' \neq b$ . Thus  $(u, z_1)$  can not be in  $V_2$ , otherwise  $z_3 = z_1$ , contradicting the fact that  $X$  is a 4-cycle. By the similar argument, we have  $V(X) \cap V_2 = \emptyset$ .
- If  $(u, z_1) \in V_3$ , then so is  $(u, z_3)$ . We first note that  $(u', z_2), (u', z_4)$  can not be in  $V_4$  because there is no edge between  $V_3$  and  $V_4$ .

If  $(u', z_2)$  or  $(u', z_4)$  is in  $V_3$ , then both are in  $V_3$ . By the definition of  $E_{3,3}$  this means that  $z_1 = z_2 = z_3 = z_4$ . It follows that  $(u, z_1) = (u, z_3)$ , which is contradiction.

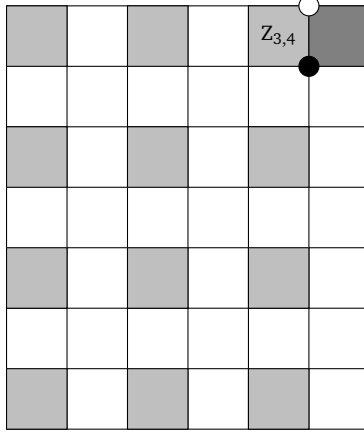


Figure 5:  $G_{7,8}$ .

So let us assume that  $(u', z_2), (u', z_4) \in V_1$ . By the definition of  $E_{13}$  and  $V_3$ , we have  $z_1 = z_3 = v_{u, z_2}$ , which leads to a contradiction.

- If  $(u, z_1) \in V_4$ , then by similar arguments, we must have  $(u, z_3) \in V_4$  and  $(u', z_2), (u', z_4) \in V_1$ . By the definition of  $E_{14}$  and since  $z_2 \neq z_4$  we have  $z_1 = z_3 = v_{u, \{z_2, z_4\}}$ . Again this leads to a contradiction.
- Finally, assume that  $V(X) \subseteq V_1$ . Thus  $u, u' \in V(G_{s,t}) \setminus (F \cup D)$  and  $uu' \in E(G_{s,t})$ . However it is easy to see that the vertices in  $V(G_{s,t}) \setminus (F \cup D)$  are mutually nonadjacent.

□

We define a further function

$$\bar{h}(i, j) := h(i, t + 1 - j).$$

for every  $i \in [s]$  and  $j \in [t]$ . The mapping  $\bar{h}$  is also an embedding from  $G_{s,t}$  to  $P$ , since  $(i, j) \mapsto (i, t + 1 - j)$  is an automorphism of  $G_{s,t}$ .

**Lemma 5.10.** *Either  $h(F_1) = C$  or  $\bar{h}(F_1) = C$ .*

*Proof.* Since  $h$  is injective and  $|C| = |F_1|$ , it suffices to prove  $C \subseteq h(F_1)$ . We consider all the 4-cycles

$$Z_{i,j} : (2i - 1, 2j - 1) \rightarrow (2i - 1, 2j) \rightarrow (2i, 2j) \rightarrow (2i, 2j - 1) \rightarrow (2i - 1, 2j - 1),$$

with  $i \in [k_1]$  and  $j \in [k_2 + 1]$ . For the grid  $G_{7,8}$  these are the light gray cycles in Figure 5. It is clear that there are  $k_1 \cdot (k_2 + 1)$  of them, and none of them share common vertices. Note  $|C| = k_1 \cdot (k_2 + 1)$ . Thus Lemma 5.9 (C1) implies that the  $h$ -image of each such cycle must contain exactly one vertex which is mapped to  $C$ , and every vertex in  $C$  is the image of one vertex in one of those cycles.

Consider the 4-cycle

$$\begin{aligned} (2k_1, 2k_2 + 1) &\rightarrow (2k_1, 2k_2 + 2) \rightarrow (2k_1 + 1, 2k_2 + 2) \\ &\rightarrow (2k_1 + 1, 2k_2 + 1) \rightarrow (2k_1, 2k_2 + 1), \end{aligned}$$

which is the gray cycle in Figure 5 for our example  $G_{7,8}$ . By Lemma 5.9 (C1) it must contain a vertex  $u$  mapped to  $C$ . As we have already argued that  $u$  is on one of the cycles  $Z_{i,j}$ , in particular  $Z_{k_1, k_2+1}$ . In addition,  $Z_{k_1, k_2+1}$  can have only one such vertex  $u$ . We conclude that either  $u = (2k_1, 2k_2 + 2)$  or  $u = (2k_1, 2k_2 + 1)$ .

Assume  $u = (2k_1, 2k_2 + 2)$ , i.e.,  $h(2k_1, 2k_2 + 2) \in C$ , as shown by the white node in Figure 5. Observe that the 4-cycle (the cycle left to  $Z_{3,4}$ )

$$\begin{aligned} (2k_1 - 2, 2k_2 + 1) &\rightarrow (2k_1 - 2, 2k_2 + 2) \rightarrow (2k_1 - 1, 2k_2 + 2) \\ &\rightarrow (2k_1 - 1, 2k_2 + 1) \rightarrow (2k_1 - 2, 2k_2 + 1) \end{aligned}$$

also must contain at least one vertex mapped to  $C$ . Using Lemma 5.9 (C2), we can conclude that it can only be  $(2k_1 - 2, 2k_2 + 2)$ . By repeating the argument inductively, we conclude that if  $h(i, j) \in C$ , then both  $i$  and  $j$  have to be even.

In the second case, we have  $u = (2k_1, 2k_2 + 1)$ , i.e.,  $h(2k_1, 2k_2 + 1) \in C$  as shown by the black node in Figure 5. Then  $\bar{h}(2k_1, 2) \in C$  by  $h(2k_1, 2k_2 + 1) \in C$ . Thus the same argument as the above case shows that if  $\bar{h}(i, j) \in C$ , then both  $i$  and  $j$  have to be even.  $\square$

Recall that our goal is to show that  $\mathcal{S}_{s,t} = (F, D)$  is rigid for  $G_{s,t}$ . In particular,  $\{(u, u) \mid u \in F\} \subseteq h(V(G_{s,t}))$ . Since  $h(V(G_{s,t})) = \bar{h}(V(G_{s,t}))$ , this is equivalent to

$$\{(u, u) \mid u \in F\} \subseteq \bar{h}(V(G_{s,t})).$$

So without loss of generality, in the following we assume that  $h(F_1) = C_1$ . That is,

$$h(i, j) \in C \iff (i, j) \in F_1. \quad (5)$$

**Lemma 5.11.**

$$\begin{aligned} &\{h(2, 2), h(2, t), h(s-1, 2), h(s-1, t)\} \\ &= \left\{ ((2, 2), (2, 2)), ((2, t), (2, t)), ((s-1, 2), (s-1, 2)), ((s-1, t), (s-1, t)) \right\}. \end{aligned}$$

*Proof.* In the grid  $G_{s,t}$  the only vertices in  $F_1$  that are connected to exactly two other vertices in  $F_1$  by paths of length 2 are  $(2, 2)$ ,  $(2, t)$ ,  $(s-1, 2)$ , and  $(s-1, t)$  (all the others are connected to either 3 or 4 vertices in  $C$ ). By Lemma 5.10,  $h$  has to map them to those vertices in  $C$  with the same property in  $H$ , which are precisely the ones in the righthand side of the above equation.  $\square$

By similar arguments as the proof of Lemma 5.3, in particular the proof of Claim 2 based on distances, and by taking automorphism if necessary, we can assume that for every  $i \in [k_1]$  and  $j \in [k_2 + 1]$

$$h(2i, 2j) = ((2i, 2j), (2i, 2j)). \quad (6)$$

**Lemma 5.12.** *For every  $u \in \{(1, 1), (s, 1), (1, t), (s, t)\}$  we have  $h(u) = (u, u)$ .*

*Proof.* We first show

$$h(1, t) = ((1, t), (1, t)).$$

Since  $h$  is an embedding and  $h(2, t) = ((2, t), (2, t))$ , it holds that  $h(2, t)$  is in

$$\left\{ ((1, t), (1, t)), ((3, t), (3, t)), ((2, t-1), (2, t-1)) \right\},$$



i.e., the set of vertices adjacent to  $((2, t), (2, t))$  in  $P$ . Assume that  $h(1, t) = ((3, t), (3, t))$ . We consider the path

$$(1, t) \rightarrow (2, t) \rightarrow (3, t) \rightarrow (4, t).$$

Under the embedding  $h$  we should get a path in  $P$  as

$$\begin{aligned} h(1, t) &= ((3, t), (3, t)) \rightarrow h(2, t) = ((2, t), (2, t)) \\ &\rightarrow h(3, t) \rightarrow h(4, t) = ((4, t), (4, t)), \end{aligned}$$

where the second and third equalities are by (6). But this clearly forces  $h(3, t) = ((3, t), (3, t)) = h(1, t)$ , which contradicts the injectivity of  $h$ . The case for  $h(2, t) = ((2, t-1), (2, t-1))$  can be similarly ruled out.

Now we proceed to show that for all  $j < t$  we have  $h(1, j) = ((1, j), (1, j))$ . Let  $j = t-1$ . Since  $(1, t)$  and  $(1, t-1)$  are adjacent in  $G$ , hence  $h(1, t) = ((1, t), (1, t))$  and  $h(1, t-1)$  are adjacent in  $P$  too. Thus,  $h(1, t-1) = ((1, t-1), (1, t-1))$ . Furthermore, using

$$h(2, t) = ((2, t), (2, t)) \quad \text{and} \quad h(2, t-2) = ((2, t-2), (2, t-2))$$

we deduce that  $h(2, t-1) = ((2, t-1), (2, t-1))$ . Combined with  $h(1, t-1) = ((1, t-1), (1, t-1))$ , we conclude that

$$h(1, t-2) = ((1, t-2), (1, t-2)).$$

Repeating the above argument, it can be reached that  $h(1, 1) = h(1, 1)$ . Similarly we can obtain that  $h(s, t) = h(s, t)$ , and finally  $h(s, 1) = h(s, 1)$ .  $\square$

Now Proposition 5.8 follows easily from Lemma 5.10 and Lemma 5.12.

*Proof of Proposition 5.1 (i).* Let  $k \in \mathbb{N}$ . Our goal is to find a graph  $G$  in  $\mathcal{K}$  with a rigid skeleton  $(F, D)$  such that  $\text{tw}((G \setminus F)/D) \geq k$ . Define

$$k^* := 2k + 4.$$

Thus there is a grid  $G_{s,t} \in \mathcal{K}$  with  $s, t \geq k^*$ . By Proposition 5.8 the skeleton  $\mathcal{S}_{s,t} = (F, D)$  in Definition 5.4 of  $G_{s,t}$  is rigid. Moreover,

$$\text{tw}((G_{s,t} \setminus F)/D) = \min \left\{ \left\lfloor \frac{s-1}{2} \right\rfloor, \left\lfloor \frac{t-2}{2} \right\rfloor \right\} - 1 \geq k,$$

by Lemma 5.7.  $\square$

## 5.2 The richness of walls

We fix some  $s > 2$  and  $t > 3$ . Let  $\mathcal{S}_{s,t}^{\text{wall}} := (F, D)$  with  $F = F_1 \cup F_2$  and  $D = \emptyset$ , where

$$\begin{aligned} F_1 &= \{v_{i,j} \mid i \in [s+1] \text{ and } j \in [2]\} \\ &\quad \cup \{u_{i,2} \mid i \in [s+1]\} \cup \{v_{i,t}, u_{i,t} \mid i \in [s+1]\} \\ &\quad \cup \{v_{i,t+1} \mid i \in [s+1] \text{ and odd } t\} \cup \{u_{i,t+1} \mid i \in [s+1] \text{ and even } t\}, \\ F_2 &= \{v_{1,j}, u_{1,j}, v_{2,j} \mid 3 \leq j < t\} \cup \{u_{s,j}, v_{s+1,j}, u_{s+1,j} \mid 3 \leq j < t\}. \end{aligned} \tag{7}$$

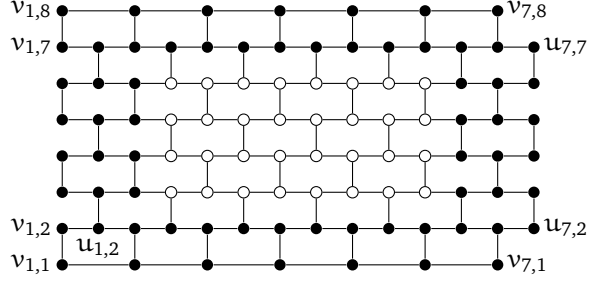


Figure 6: A skeleton  $(F, D)$  for  $W_{6,7}$ , where  $F$  is the set of black vertices and  $D = \emptyset$ .

That is, we take as the set  $F$  the bottom two rows, the top two rows, and the leftmost three columns, and the rightmost three columns of the vertices in  $W_{s,t}$ . Figure 6 shows the case for  $W_{6,7}$ .

The following observation is straightforward.

**Lemma 5.13.** *Every 5-cycle in  $W_{s,t}$  is contained in  $F_1 \subseteq F$ .*

Clearly those 5-cycles are shortest odd cycles in  $W_{s,t}$ . The next lemma explains their importance. Recall that a *closed walk* in a graph  $G$  is a sequence

$$v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k \rightarrow v_{k+1} = v_1$$

of vertices such that  $v_i v_{i+1} \in E(G)$  for every  $i \in [k]$ . Its length is  $k$ . Thus, a  $k$ -cycle is a closed walk of length  $k$  with  $k \geq 3$  and  $v_i \neq v_j$  for every  $1 \leq i < j \leq k$ .

**Lemma 5.14.** *Let  $G$  be a graph and  $k$  the length of a shortest odd cycle in  $G$ . Then every closed walk in  $G$  of length  $k$  has to be a  $k$ -cycle.*

*Proof.* Let

$$Z : v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k \rightarrow v_1 \tag{8}$$

be a closed walk in  $G$ . We need to show that  $v_i \neq v_j$  for every  $1 \leq i < j \leq k$ . Assume otherwise, choose such  $i, j$  with  $v_i = v_j$  and all  $v_k$ 's in between pairwise distinct. Then either  $j = i + 2$  or

$$Z_1 : v_i \rightarrow v_{i+1} \rightarrow v_{i+2} \rightarrow \cdots \rightarrow v_j$$

is a cycle. Note that  $Z_1$  must be an even cycle, as no odd cycle in  $G$  has length smaller than  $k$ . Thus in both cases, we can shorten the closed walk (8) to

$$v_1 \rightarrow \cdots \rightarrow v_i \rightarrow v_{j+1} \rightarrow \cdots \rightarrow v_1$$

which is still a closed walk of odd length. By repeating this procedure, eventually we obtain a cycle of odd length in  $G$ , which contradicts the minimality of  $k$ .  $\square$

**Corollary 5.15.** *Let  $G$  be a graph and  $k$  the length of a shortest odd cycle in  $G$ . Then every endomorphism of  $G$  maps every  $k$ -cycle to a  $k$ -cycle.*

*Proof.* This is immediate by observing that every homomorphism maps a cycle to a walk and by Lemma 5.14.  $\square$

**Lemma 5.16.** *The set  $F$  defined in (7) is a frame for the wall  $W_{s,t}$ .*

*Proof.* We discuss the cases as exemplified in Figure 6 where  $s \leq t$  and  $t \geq 5$  is odd. The others can be argued in a similar fashion.

Let  $h$  be an endomorphism of  $W_{s,t}$  with  $F \subseteq h(V(W_{s,t}))$ . We consider four pair of vertices

$$(v_{1,2}, v_{s+1,t}), (v_{1,2}, u_{s+1,t}) \quad \text{and} \quad (v_{1,t}, v_{s+1,2}), (v_{1,t}, u_{s+1,2}).$$

It is not hard to see that they have the largest distance in  $W_{s,t}$ , and the distance between any other pair of vertices is strictly smaller. Figure 7 illustrates the situation for  $W_{6,7}$ . We proceed similarly as Claim 1 in the proof of Lemma 5.3 to obtain

$$\begin{aligned} h(\{v_{1,2}, v_{s+1,t}, u_{s+1,t}\}) &= \{v_{1,2}, v_{s+1,t}, u_{s+1,t}\} \\ \text{and } h(\{v_{1,t}, v_{s+1,2}, u_{s+1,2}\}) &= \{v_{1,t}, v_{s+1,2}, u_{s+1,2}\}, \end{aligned}$$

or

$$\begin{aligned} h(\{v_{1,2}, v_{s+1,t}, u_{s+1,t}\}) &= \{v_{1,t}, v_{s+1,2}, u_{s+1,2}\} \\ \text{and } h(\{v_{1,t}, v_{s+1,2}, u_{s+1,2}\}) &= \{v_{1,2}, v_{s+1,t}, u_{s+1,t}\}. \end{aligned}$$

Note that  $v_{1,2}$ ,  $v_{s+1,2}$ ,  $v_{1,t}$  and  $v_{s+1,t}$  are on 5-cycles, while  $u_{s+1,2}$  and  $u_{s+1,t}$  are not. Corollary 5.15 implies that  $h$  maps every 5-cycle in  $W_{s,t}$  to a 5-cycle. Thus we can assume without loss of generality that

$$h(v_{1,2}) = v_{1,2} \quad \text{and} \quad h(v_{1,t}) = v_{1,t},$$

and hence  $h(w) = w$  for any  $w \in \{v_{s+1,2}, u_{s+1,2}, v_{s+1,t}, u_{s+1,t}\}$ . Once these six vertices are in the right place, by an easy induction we can conclude that  $h(w) = w$  for every  $w \in F_1$ .

Then we proceed to show  $h(w) = w$  for every  $w \in F_2$ . As a first step, consider the possibility of  $h(u_{1,3})$ . Note that  $h(u_{1,2}) = u_{1,2}$  (for  $u_{1,2} \in F_1$ ) implies

$$h(u_{1,3}) \in \{v_{1,2}, v_{2,2}, u_{1,3}\}. \tag{9}$$

On the other hand,

$$\text{dist}^{W_{s,t}}(u_{1,3}, v_{1,t}) \geq \text{dist}^{W_{s,t}}(h(u_{1,3}), h(v_{1,t})) = \text{dist}^{W_{s,t}}(h(u_{1,3}), v_{1,t}).$$

With (9) we can conclude  $h(u_{1,3}) = u_{1,3}$ .

Recall  $v_{1,3} \in F_2 \subseteq h(V(W_{s,t}))$ . Choose an arbitrary  $w \in V(W_{s,t})$  with  $h(w) = v_{1,3}$ . We observe that for every  $z \in \{v_{1,2}, u_{s+1,2}, v_{1,t}, u_{s+1,t}\} \subseteq F_1$

$$\text{dist}^{W_{s,t}}(v_{1,3}, z) = \text{dist}^{W_{s,t}}(h(w), h(z)) \leq \text{dist}^{W_{s,t}}(w, z).$$

But this implies  $w = v_{1,3}$ .

For the remaining vertices in  $F_2$ , we can argue similarly. And finally by observing the distance between any vertex in  $V(W_{s,t}) \setminus (F_1 \cup F_2)$  and  $F_1 \cup F_2$ , we can establish the surjectivity of  $h$ .  $\square$

**Lemma 5.17.** *Assume:*

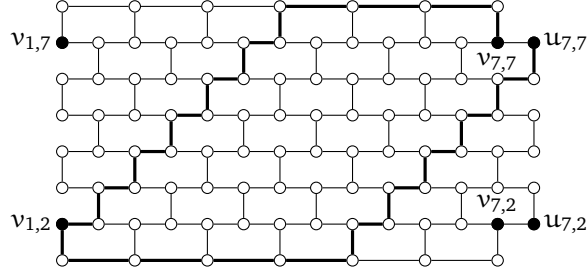


Figure 7: Two shortest paths between  $v_{1,2}$  and  $v_{7,7}$  and between  $v_{1,2}$  and  $u_{7,7}$  are shown in black.

- $G$  is a graph with a skeleton  $\mathcal{S} = (F, D)$ .
- $H$  is a graph with a coloring  $\chi : V(H) \rightarrow V(G) \setminus (F \cup D)$ .
- $P = P(G, \mathcal{S}, H, \chi)$  is the product graph as defined in Definition 4.5.
- $k$  is the length of a shortest odd cycle in  $G$ .

Then for every cycle  $Z$  in  $P$  of length  $k$ , the set

$$\{v \in V(G) \mid (v, a) \text{ occurs in } Z \text{ for some } a\}$$

induces a cycle in  $G$  of length  $k$ .

*Proof.* Assume  $Z : (v_1, a_1) \rightarrow (v_2, a_2) \rightarrow \dots \rightarrow (v_k, a_k) \rightarrow (v_1, a_1)$ . By Lemma 4.8 we conclude that

$$\pi_1(Z) : v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k \rightarrow v_1 \quad (10)$$

is a closed walk in  $G$  of length  $k$ . Then the result follows immediately from Lemma 5.14.  $\square$

**Proposition 5.18.**  $\mathcal{S}_{s,t}^{\text{wall}}$  is a rigid skeleton of  $W_{s,t}$ .

*Proof.* Let  $H$  be a graph and  $\chi : V(H) \rightarrow V(W_{s,t}) \setminus F$  where  $\mathcal{S}_{s,t}^{\text{wall}} = (F, \emptyset)$ . Moreover let  $P = P(W_{s,t}, \mathcal{S}_{s,t}^{\text{wall}}, H, \chi)$  and  $h$  be an embedding from  $W_{s,t}$  to  $P$ . We need to show that

$$\{(u, u) \mid u \in F\} \subseteq h(V(W_{s,t})). \quad (11)$$

Again those 5-cycles  $Z$  in  $W_{s,t}$  plays a vital role. Since  $h$  is an embedding, the image  $h(Z)$  has to be a 5-cycle in  $P$ . Hence Lemma 5.17 implies that  $\pi_1(h(Z))$  remains a 5-cycle in  $W_{s,t}$ . Then  $h(F_1) = \{(u, u) \mid u \in F_1\}$  by an easy counting, and (11) follows again by observing the distance between  $F_1$  and  $F_2$ .  $\square$

Now the remaining part of the proof of the richness of walls (i.e., Proposition 5.1 (ii)) is easy, and thus left to the reader.

## 6 Conclusions

We have shown that the parameterized embedding problem on the classes of all grids and all walls is hard for  $W[1]$ . Our proof exploits some general structures in those graphs, i.e., frames and skeletons, thus is more generic than other known  $W[1]$ -hard cases. We expect that our machinery can be used to solve some other cases. However, it could be seen that the class of complete bipartite graphs is not rich. Hence the result of [15] is not a special case of our Theorem 4.15. Resolving the **Dichotomy Conjecture** for the embedding problem might require a unified understanding of the cases of biclique and grids.

A remarkable phenomenon of the homomorphism problem is that the polynomial time decidability of  $\text{HOM}(\mathcal{K})$  coincides with the fixed-parameter tractability of  $p\text{-HOM}(\mathcal{K})$  for any class  $\mathcal{K}$  of graphs [13], assuming  $\text{FPT} \neq W[1]$ . For the embedding problem this is certainly not true, as for the class  $\mathcal{K}$  of all paths  $\text{EMB}(\mathcal{K})$  is NP-hard, yet  $p\text{-EMB}(\mathcal{K}) \in \text{FPT}$ . Thus, in the **Dichotomy Conjecture**, the tractable side is really in terms of fixed-parameter tractability. But it is still interesting and important to give a precise characterization of those  $\mathcal{K}$  whose  $\text{EMB}(\mathcal{K})$  are solvable in polynomial time. At the moment, we don't even have a good conjecture.

## References

- [1] N. Alon, R. Yuster, and U. Zwick. Color-coding. *Journal of the ACM*, 42(4):844–856, 1995.
- [2] M. Bodirsky and M. Grohe. Non-dichotomies in constraint satisfaction complexity. In *Automata, Languages and Programming, 35th International Colloquium, ICALP 2008, Reykjavik, Iceland, July 7-11, 2008, Proceedings, Part II - Track B: Logic, Semantics, and Theory of Programming & Track C: Security and Cryptography Foundations*, pages 184–196, 2008.
- [3] H. Chen and M. Müller. One hierarchy spawns another: Graph deconstructions and the complexity classification of conjunctive queries. In *Proceedings of the Joint Meeting of the 23rd EACSL Annual Conference on Computer Science Logic and the 29th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 32:1–32:10, 2014.
- [4] Y. Chen, M. Thurley, and M. Weyer. Understanding the complexity of induced subgraph isomorphisms. In *Automata, Languages and Programming, 35th International Colloquium, ICALP 2008, Reykjavik, Iceland, July 7-11, 2008, Proceedings, Part I: Track A: Algorithms, Automata, Complexity, and Games*, pages 587–596, 2008.
- [5] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.
- [6] V. Dalmau, P. G. Kolaitis, and M. Y. Vardi. Constraint satisfaction, bounded treewidth, and finite-variable logics. In *Principles and Practice of Constraint Programming - CP 2002, 8th International Conference, CP 2002, Ithaca, NY, USA, September 9-13, 2002, Proceedings*, pages 310–326, 2002.
- [7] R. Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

- [8] R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Springer-Verlag, 1999.
- [9] R.G. Downey and M.R. Fellows. Fixed-parameter tractability and completeness II: On completeness for  $W[1]$ . *Theoretical Computer Science*, 141:109–131, 1995.
- [10] D. Eppstein. Subgraph isomorphism in planar graphs and related problems. *Journal of Graph Algorithms and Applications*, 3:1–27, 1999.
- [11] J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer, 2006.
- [12] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Freeman, 1979.
- [13] M. Grohe. The complexity of homomorphism and constraint satisfaction problems seen from the other side. *Journal of the ACM*, 54(1):1:1–1:24, 2007.
- [14] M. Grohe, T. Schwentick, and L. Segoufin. When is the evaluation of conjunctive queries tractable. In *Proceedings of the 33rd ACM Symposium on Theory of Computing*, pages 657–666, 2001.
- [15] B. Lin. The parameterized complexity of  $k$ -biclique. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015*, pages 605–615, 2015.
- [16] D. Marx and M. Pilipczuk. Everything you always wanted to know about the parameterized complexity of subgraph isomorphism (but were afraid to ask). *ArXiv (CoRR)*, abs/1307.2187, 2013.
- [17] D.W. Matula. Subtree isomorphism in  $o(n^{5/2})$ . In P. H. B. Alspach and D. Miller, editors, *Algorithmic Aspects of Combinatorics*, volume 2 of *Annals of Discrete Mathematics*, pages 91–106. Elsevier, 1978.
- [18] B. Monien. How to find longest paths efficiently. In *Analysis and design of algorithms for combinatorial problems*, volume 109 of *North Holland Mathematics Studies*, pages 239–254. North Holland, 1985.
- [19] J. Plehn and B. Voigt. Finding minimally weighted subgraphs. In R. Möhring, editor, *Graph-Theoretic Concepts in Computer Science, WG '90*, volume 484 of *Lecture Notes in Computer Science*, pages 18–29. Springer Verlag, 1990.
- [20] N. Robertson and P.D. Seymour. Graph minors V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41:92–114, 1986.
- [21] J.R. Ullman. An algorithm for subgraph isomorphism. *Journal of the ACM*, 23(1):31–42, 1976.